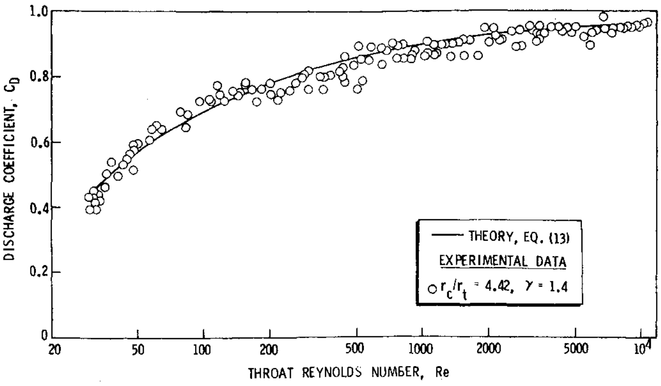

Fig. 4 Discharge coefficient—comparison of theory [Eq. (13)] and experimental data from Ref. 2.

of velocity slip on the discharge coefficient is significant at throat Reynolds numbers less than about 10^3. The effect of curvature becomes important at Reynolds numbers on the order of 2×10^2.

Equation (13) is intended to be used for predicting discharge coefficients for supersonic nozzles for $0 \leq (r_c/r_t) \leq 20$ and $50 < Re < 10^5$. At throat Reynolds numbers greater than 10^5, the boundary layer at the throat is turbulent so that the laminar theory presented in this paper is no longer applicable. At throat Reynolds numbers less than 50, the boundary layer may fill the entire throat. For the velocity and density profiles used in this study, this occurs at $C_D \simeq 0.4$ (when $u_s = 0$). When the boundary layer fills the entire throat, the flow may not necessarily choke (e.g., Fig. 1, Ref. 2) and the discharge coefficient may be strongly dependent on the back pressure. Thus, for these extremely low Reynolds numbers Eq. (13) is no longer applicable.

References

[1] Smetana, F. O., "A Semi-empirical Description of the Discharge Characteristics of the Converging Section of a Low Density Hypersonic Nozzle," *Journal of the Aerospace Sciences*, Vol. 28, No. 12, 1961, p. 988.

[2] Smetana, F. O., "Convergent-Divergent Nozzle Discharge Characteristics in the Transition Regime Between Free Molecule and Continuum Flow," ASME Paper 63-WA-94, New York, 1963.

[3] Folsom, R. G., "Nozzle Characteristics in High Vacuum Flows—Rarefied Gas Dynamics," *ASME Transactions*, Vol. 74, 1952, p. 915.

[4] Milligan, M. W., "Nozzle Characteristics in the Transition Regime Between Continuum and Free Molecular Flow," *AIAA Journal*, Vol. 2, No. 6, June 1964, pp. 1088–1092.

[5] John, R. R., "Resistojet Research and Development—Phase II (Final Report)," NASA-CR-54688, AVSSP-0356-66-CR, Dec. 1966, Avco, Wilmington, Mass.

[6] Lefkowitz, B., "A Study of Rotational Relaxation in a Low-Density Hypersonic Freejet by Means of Impact-Pressure Measurement," Rept. 67-27, July 1967, Ph.D. dissertation, Univ. of California at Los Angeles.

[7] Rae, W. J., "Some Numerical Results on Viscous Low-Density Nozzle Flows in the Slender-Channel Approximation," AIAA Paper 69-654, San Francisco, Calif., 1969.

[8] Bartz, D. R., "Turbulent Boundary Layer Heat Transfer From Rapidly Accelerating Flow of Rocket Combustion Gases and of Heated Air," *Advances in Heat Transfer*, Vol. 2, Academic Press, New York, 1965.

[9] Schlichting, H., *Boundary Layer Theory*, 4th ed., McGraw-Hill, New York, 1960.

[10] Oswatitsch, K. and Kuerti, G., *Gas Dynamics*, Academic Press, New York, 1956.

[11] Back, L. H., Massier, P. F., and Gier, H. L., "Comparison of Measured and Predicted Flows Through Conical Supersonic Nozzle, With Emphasis on the Transonic Region," *AIAA Journal*, Vol. 3, No. 9, Sept. 1965, pp. 1606–1614.

[12] Durham, F. P. and Wood, K. D., "The Performance Characteristics of Small Rocket-Type Nozzles," Repts. 1–6, Feb. 1953–June 1955, Engineering Experiment Station, Univ. of Colorado, Boulder, Colo.

[13] "Standards for Discharge Measurement With Standardized Nozzle and Orifices," translated from *German Industrial Standard*, 1952, 4th ed., TM-952, Sept. 1940, NACA.

[14] *Fluid Meters, Their Theory and Application*, 4th edition, ASME, New York, 1937.

[15] *Flowmeter Computation Handbook*, ASME, New York, 1961.

Sharp Slender Cones in Near-Free-Molecule Hypersonic Flow

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IN the near-free-molecule-flow regime, only limited data drag for slender cones are currently available.[1,2] This Note presents additional drag measurements in this flow regime, obtained in air as well as helium. These results, which extend the data of Refs. 1 and 2 to higher Knudsen numbers, were obtained for cones with half angles from 2.5° to 10° at Mach numbers of 24 and 27 for air and 35 for helium. The Knudsen number based on cone diameter (λ_∞/D) varied from 0.01 to 5.

The data were obtained in the Ames 42-Inch Shock Tunnel. The general operation and calibration procedures of this facility using a combustion driver are described elsewhere.[3,4] For completeness, a short discussion of how the freestream properties were determined for these tests will be given. The stream properties for the air tests were obtained from static and impact pressure measurements by a method[4] that assumes the air to be in equilibrium from the reservoir to an arbitrary point in the nozzle where chemical reactions and molecular vibrations are thereafter frozen.[5] At the present test conditions (M_∞ = 24 and 27), static pressure measurements in the test section are unreliable because of the large corrections necessary to account for low-density effects. Therefore, the freeze Mach number for each test condition was determined using static and impact pressure measurements taken upstream in the conical nozzle where corrections for low-density effects on the static pressure probe were less than 10%. The freestream properties in the test section were then obtained using the upstream freeze Mach number, an impact pressure measurement in the test section, and a one-dimensional nozzle expansion computer program. To insure that the impact pressure measurement at the test station was free from rarefaction effects, probes of several diam. (0.5 to 4 cm) were used. The measured results indicated that these effects were negligible for probe diameters greater than 1.5 cm. The accuracy of the measured run-to-run variation of normalized dynamic pressure was ±5%; other stream properties as derived from computations of an expanding frozen flow of known active energy are estimated to be within ±10%.

With helium as a test gas the freestream properties were readily defined, because at the low reservoir enthalpy (2.4 KJ/gm), helium acts as a perfect gas in equilibrium. Therefore, measurements of the freestream impact pressure, reservoir total pressure, and total enthalpy completely specified the freestream properties. The accuracy of these properties, including run-to-run variation of normalized dynamic pressure, was within ±5%.

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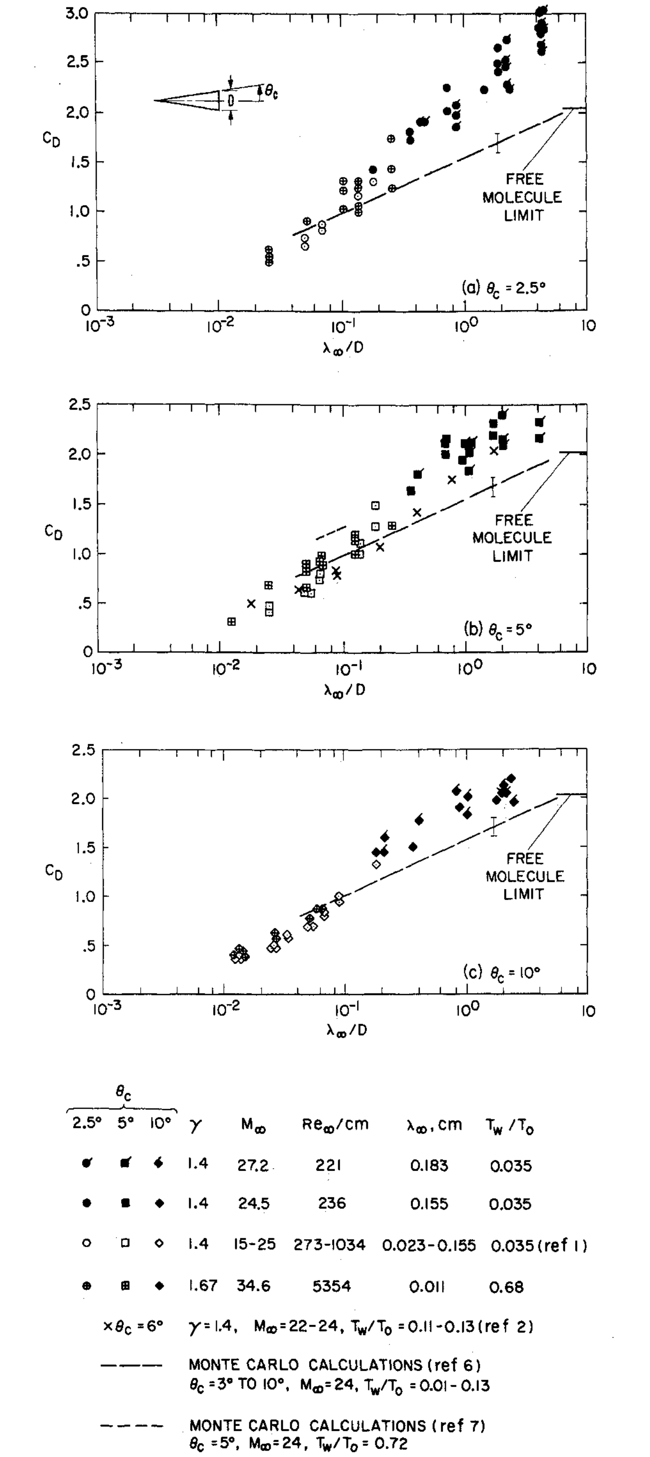

Fig. 1 Variation of cone-drag coefficient with Knudsen number.

The nominal test conditions are given in Table 1. For all tests, the model wall temperatures were 294°K. The freestream mean free path was determined from the hard sphere collision model (i.e., $\lambda_\infty = 1.26\ (\gamma)^{1/2}M_\infty/\mathrm{Re}_\infty$).

For these tests, a free-flight test technique and data reduction scheme were used that was similar to that described in Ref. 1, except for the launch procedure, which employed a retractable table like the one described in Ref. 2. The drag force data were normalized by the tunnel dynamic pressure, the latter obtained directly from the Pitot pressure measurement discussed previously. The accuracy of the measured data was calculated to be ±10%.

The present drag results in air and helium are shown in Fig. 1 for θ_c from 2.5° to 10°, along with previously published[1,2] data. For reference, the free molecule limits assuming diffuse reflection and unit thermal accommodation are also shown. Predictions by the Monte Carlo simulation method[6,7] are presented and will be discussed later. In air, as λ_∞/D increased, significant differences in drag were measured between the 2.5° cone and higher angle cones. At high λ_∞/D ($\lambda_\infty/D > 2$) the drag of the 2.5° cone is significantly above the free molecular drag (Fig. 1a). At low Mach numbers, Sims[8] experimentally obtained C_D on slender cones somewhat above the free molecule value. However, to the authors' knowledge, this is the first time this "overshoot" has been experimentally observed for cones in hypersonic flow. At $(\lambda_\infty/D) > 1$, the 10° cone drag data approach the free molecular limit, while the 5° data (on the average) fall about 10% above this limit. Over the range of λ_∞/D where both air and helium data were obtained, there is agreement, within the experimental accuracy, between the hot-wall helium and cold-wall air results. Earlier drag data[1,2] agree with the present results.

Table 1 Nominal running conditions for the 42-in. shock tunnel

γ	M_∞	P_o, atms	Re_∞/cm	Enthalpy, KJ/gM	T_w/T_o	λ_∞, cm
1.4	27.2	285	221	9.2	0.035	0.183
1.4	24.5	285	236	9.2	0.035	0.155
1.67	34.6	8.6	5354	2.4	0.68	0.011

Available Monte Carlo calculations[6,7] for hard sphere molecules are shown in Fig. 1. When plotted in the present form, C_D vs λ_∞/D, all computed points for θ_c = 3°,6°,8°, and 10° fall within the indicated scatter bar. These calculations agree well with both the air and helium data for $\lambda_\infty/D < 10^{-1}$. For θ_c = 5°, where both hot-wall and cold-wall[6] calculations are available, the hot wall helium data indicate better agreement with the cold-wall calculations. At $\lambda_\infty/D > 10^{-1}$, the calculations underpredict the air data. At high λ_∞/D the difference between theory and experiment may be due to differences in the diatomic molecular collision process and the hard sphere model used by theory or the surface reflection and energy accommodation laws. (The calculations assume diffuse reflection and complete energy accommodation.) However, at hypersonic speeds over cold-wall test models, the hard sphere assumption should be valid for air over the first few local mean free paths of the cone, which for the present test conditions would include the entire cone length. Thus, the difference between theory and experiment is perhaps due to the surface interaction laws. This may explain the agreement between the calculations and the data at low Knudsen numbers, since surface interaction laws probably diminish in importance at the lower Knudsen numbers. Although many surface interaction laws have been proposed (e.g., Ref. 9), there are at present no Monte Carlo calculations for cones that use other than the diffuse reflection and complete energy accommodation laws mentioned previously. Evidently, a valid test of this simulation technique remains dependent on the surface interaction laws and perhaps the diatomic molecular collision interactions that currently are unresolved.

References

[1] Kussoy, M. I. and Horstman, C. C., "Cone Drag in Rarefied Hypersonic Flow," *AIAA Journal*, Vol. 8, No. 2, Feb. 1970, p. 315.

[2] Geiger, R. E., "Slender Cone, Zero Angle of Attack Drag in Continuum and Noncontinuum Flow," AIAA Paper 69-711, San Francisco, Calif., June, 1969.

[3] Loubsky, W. J., Hiers, R. S., and Stewart, D. A., "Performance of a Combustion Driven Shock Tunnel With Application to the Tailored-Interface Operating Conditions," *Proceedings of the Third Conference on Performance of High Temperature Systems*, TMX 54960, Dec. 7–9, 1964, NASA.

[4] Hiers, R. S., Jr. and Reller, J. O., Jr., "Analysis of Non-equilibrium Air Streams in the Ames 1-Foot Shock Tunnel," TN D-4985, 1969, NASA.
[5] Yoshikawa, Kenneth K. and Katzen, Elliott D., "Charts for Air-Flow Properties in Equilibrium and Frozen Flows in Hypervelocity Nozzles," TN D-693, 1961, NASA.
[6] Vogenitz, F. W. and Takata, G. Y., "Rarefied Hypersonic Flow About Cones and Flat Plates," *AIAA Journal*, Vol. 9, No. 1, Jan. 1971, p. 94.
[7] Vogenitz, F. W., private communication, Jan. 1970.
[8] Sims, W. H., "Drag Coefficients of Sharp and Blunt Cones in Highly Rarefied Supersonic Flow," *Rarefied Gas Dynamics*, Academic Press, New York, 1969, Vol. 1, p. 575.
[9] Hurlbut, F. C. and Sherman, F. S., "Application of the Nocilla Wall Reflection Model to Free-Molecule Kinetic Theory," *The Physics of Fluids*, Vol. 11, No. 3, March 1968, p. 486.

Stability Test for a Related Routh-Hurwitz Problem

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Introduction

The necessary and sufficient conditions for the asymptotic stability of a system described by a set of linear ordinary differential equations with constant coefficients are given by the well-known Routh-Hurwitz criterion or by the existence of a quadratic Lyapunov function. Corresponding conditions for the stability (i.e., none of the characteristic roots have positive real parts, some may be distinct pure imaginaries) are sometimes of interest. It is true for practical systems that asymptotic stability is a much more important requirement than is stability. But the latter concept often arises in the idealized modeling of physical systems, for instance, in the rotation of a torque-free rigid body, and in the classical theory of the libration of the moon.

Necessary and sufficient conditions for stability may be obtained by an extension of the method used in the derivation of the Routh-Hurwitz criterion. Certain theorems in this connection are given in Lehnigk,[1] although the method is not well-documented. The purpose of the present article is to outline a step-by-step stability test procedure, utilizing simplifications provided by some recent results.

Consider a dynamic system with the following real characteristic equation

$$f(x) = x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots + a_n = 0 \tag{1}$$

If $a_n = 0$, Eq. (1) has at least a zero root, and can be reduced to a lower order system immediately. Therefore, assume $a_n = 0$ for convenience. Also define as special roots of Eq. (1) any roots $x = \pm x^*$.

It is useful to decompose $f(x)$ in the following two ways:

$$\begin{aligned} 1)\quad f(x) &= h(x^2) + xg(x^2) \\ h(x^2) &= a_n + a_{n-2}x^2 + \ldots \qquad (2) \\ g(x^2) &= a_{n-1} + a_{n-3}x^2 + \ldots \\ 2)\quad f(x) &= p(x)\,q(x) = p(x)s(x^2) \end{aligned}$$

where $p(x)$ is a polynomial without any special roots and $q(x) = s(x^2)$ is a polynomial with special roots only. If $f(x)$ has any pure imaginary roots, $q(x)$ will not be an identical constant. The decomposition Eq. (3) is the crux of the problem.

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Once this is accomplished, it follows the system; Eq. (1) is stable if and only if all roots of $p(x)$ have negative real parts and all roots of $q(x)$ are distinct pure imaginaries, or equivalently, $s(y)$ has only distinct negative roots. With these as preliminaries, a step-by-step stability test procedure is outlined in the following section.

A Step-by-Step Stability Test

1) If $f(x)$ has only special roots, i.e., if it takes on the form $s(x^2)$, go to step 5 below.

2) If $f(x)$ is not of the form $s(x^2)$, then if any of the coefficients a_i $(i = 1,2,\ldots n)$ is negative or zero, the system is unstable, no need to proceed further.

3) Define, as usual, Hurwitz determinants $\Delta_1, \Delta_2, \ldots \Delta_{n-1}$ as the successive principal minors of the following $(n \times n)$ Hurwitz matrix:

$$\begin{bmatrix} a_1 & a_3 & a_5 & \cdot & 0 & 0 \\ 1 & a_2 & a_4 & \cdot & 0 & 0 \\ 0 & a_1 & a_3 & \cdot & 0 & 0 \\ 0 & 1 & a_2 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdot & a_{n-2} & a_n \end{bmatrix}$$

If n is even, evaluate the Hurwitz determinants $\Delta_3, \Delta_5, \ldots \Delta_{n-1}$. If n is odd, evaluate the Hurwitz determinants $\Delta_2, \Delta_4, \ldots \Delta_{n-1}$. a) If any one of these Hurwitz determinants is negative, the system is unstable, no need to proceed further. b) If all Hurwitz determinants are positive, the system is asymptotically stable, no need to proceed further.

4) If the Hurwitz determinants satisfy the following condition,

$$\Delta_{n-1} = \Delta_{n-3} = \ldots = \Delta_{n-2r+1} = 0, \qquad \Delta_{n-2r-1} \neq 0 \tag{4}$$

the system can still possible be stable, but not asymptotically stable. Now $p(x)$ must be a polynomial of degree $n - 2r$, and the Hurwitz determinants of $p(x)$ are identical to Δ_{n-m}, $m = 2r, 2r + 1, \ldots$. It follows: a) if in addition to the conditions of Eq. (4), $\Delta_{n-2t-1} = 0$ for some $t > r$, then $p(x)$ has some root with a positive real part and the system is unstable, no need to proceed further. b) If in addition to conditions (4), $\Delta_{n-2t-1} > 0$ for all $t > r$, then $s(y = x^2) = y^r + b_1y^{r-1} + b_2y^{r-2} \ldots + b_r$ is obtainable as follows. Replace the last element of the last row of the Hurwitz determinant Δ_{n-2r-1} by $g(y)$, the element above this by $h(y)$ the next element above by $yg(y)$, the next element above by $y^2g(y)$, etc. The resulting determinant is equal to $s(y)$, multiplied by a constant.

a) If any of the coefficients b_l $(l = 1,2\ldots r)$ is negative or zero, $q(x)$ has some root with positive real part, the system is unstable, no need to proceed further. b) Form the polynomial

$$s(y^2) + ys'(y^2) = y^{2r} + b_1y^{2(r-1)} + b_2y^{2(r-2)} + \ldots + b_r + ry^{2r-1} + b_1(r-1)y^{2r-3} + b_2(r-2)y^{2r-5} + \ldots + b_{r-1}y$$

together with its associated $(2r + 2r)$ Hurwitz matrix

$$\begin{bmatrix} r & b_1(r-1) & b_2(r-2) & \cdot & 0 & 0 \\ 1 & b_1 & b_2 & \cdot & 0 & 0 \\ 0 & r & b_1(r-1) & \cdot & 0 & 0 \\ 0 & 1 & b_1 & \cdot & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdot & b_{r-1} & 0 \\ 0 & 0 & 0 & \cdot & b_{r-1} & b_r \end{bmatrix}$$

and evaluate the corresponding Hurwitz determinants. If all the Hurwitz determinants $\Delta_3 = b_1^2(r-1) - 2b_2r$, Δ_5, Δ_7, $\ldots \Delta_{2r-1}$ are positive $q(x)$ has only distinct pure imaginary